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ESTIMATES FOR COEFFICIENTS OF UNIVALENT FUNCTIONS FROM INTEGRAL MEANS AND GRUNSKY INEQUALITIES

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ABSTRACT

Sharp bounds are obtained for the coefficients of inverses of univalent functions in the class $\Sigma(p)$ by using results on integral means and generalized Grunsky inequalities. A new and elementary proof is given for a result due to Löwner about sharp bounds for coefficients of inverses of functions in the class S.

1. Introduction

Let Σ denote the family of univalent functions

$$g(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}$$

in $\tilde{\Delta} = \{\zeta : 1 < |\zeta| < \infty\}$, the exterior of the unit disk Δ in the complex plane C. If $\zeta = G(\omega)$ is the inverse of a function $g \in \Sigma$, then G has an expansion

$$G(\omega) = \omega + \sum_{n=0}^{\infty} B_n \omega^{-n} \tag{1.1}$$

near $\omega = \infty$. Using a variational method, Netanyahu [Ne] obtained the following sharp estimates.

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1.2. THEOREM ([Ne, Theorem, p.339]): *Let $g \in \Sigma$ and $g(\zeta) \neq 0$. If the inverse of g has expansion* (1.1) *near ∞, then*

$$|B_0| \leq 2 \text{ and } |B_n| \leq \frac{(2n)!}{n!(n+1)!} \tag{1.3}$$

for $n = 1, 2, \ldots$. The equality holds in (1.3) *for a single n only if*

$$g(\zeta) = \zeta + 2e^{i\beta} + e^{2i\beta}\zeta^{-1}$$

for some real β.

In Netanyahu's result, the assumption that $g(\zeta) \neq 0$ is crucial. This naturally leads to the following question. What happens if $g \in \Sigma$ vanishes at some point? The purpose of this paper is to estimate the inverse coefficients of $g \in \Sigma$ when g vanishes at some point in $\tilde{\Delta}$ by using the integral mean method (see [Ba] and [BS]) and by using some generalized Grunsky inequalities (see [Sc] and [Re]). In this paper we also give a new and elementary proof for a result due to Löwner [Lö] about the inverse coefficients of univalent functions in the well known class S.

2. Integral mean and generalized Grunsky inequalities

For $1 < p < \infty$ let $\Sigma(p)$ denote the class of all meromorphic univalent functions $g \in \Sigma$ with $g(-p) = 0$. Next for $0 < p < 1$ let $S(p)$ denote the class of all meromorphic univalent functions $f(z)$ in $\Delta = \{z : |z| < 1\}$ with the normalization $f(0) = 0$, $f'(0) = 1$, and $f(-p) = \infty$. By using a powerful method introduced by Baernstein [Ba], Kirwan and Schober [KS] established the following integral mean inequality.

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{\alpha} d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |k_p(\frac{1}{re^{i\theta}})|^{-\alpha} d\theta \tag{2.1}$$

for all $f \in S(p)$, $0 < r < 1$, and $-\infty < \alpha < \infty$, where

$$k_p(\zeta) = \zeta + (p + p^{-1}) + \zeta^{-1} \tag{2.2}$$

for $\zeta = z^{-1} \in \tilde{\Delta}$. Using (2.1) and the relation that

$$g(\zeta) \in \Sigma(p) \Longleftrightarrow f(z) = g(1/z)^{-1} \in S(1/p), \tag{2.3}$$

one can easily derive that for $g(\zeta) \in \Sigma(p)$,

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^\alpha d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |k_p(re^{i\theta})|^\alpha d\theta \tag{2.4}$$

for all $1 < r < \infty$ and $-\infty < \alpha < \infty$.

Another preliminary result we need is the integral form of an exponentiated Grunsky inequality. It is well known that for $g \in \Sigma$ we have the following generalized Grunsky inequality (see, for example, [Du, Theorem 4.3])

$$\begin{aligned} &\Big|\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\beta_k \log\frac{z_j-\zeta_k}{g(z_j)-g(\zeta_k)}\Big|^2 \\ &\qquad\leq \sum_{j,s=1}^{n}\alpha_j\bar{\alpha}_s \log\frac{1}{1-(z_j\bar{z}_s)^{-1}} \sum_{k,t=1}^{n}\beta_k\bar{\beta}_t \log\frac{1}{1-(\zeta_k\bar{\zeta}_t)^{-1}} \end{aligned}$$

whenever $n \geq 1$ is an integer, $\{\alpha_j\}, \{\beta_k\} \subset C$ and $\{z_j\}, \{\zeta_k\} \subset \tilde{\Delta}$. It is also known that this inequality can be exponentiated to the following form (see, for example, [Re, 2.37]). For $g \in \Sigma$,

$$\begin{aligned} &\Big|\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\beta_k \frac{z_j-\zeta_k}{g(z_j)-g(\zeta_k)}\Big|^2 \\ &\qquad\leq \sum_{j,s=1}^{n}\alpha_j\bar{\alpha}_s \frac{1}{1-(z_j\bar{z}_s)^{-1}} \sum_{k,t=1}^{n}\beta_k\bar{\beta}_t \frac{1}{1-(\zeta_k\bar{\zeta}_t)^{-1}} \end{aligned} \tag{2.5}$$

whenever $n \geq 1$ is an integer, $\{\alpha_j\}, \{\beta_k\} \subset C$ and $\{z_j\}, \{\zeta_k\} \subset \tilde{\Delta}$.

Now we transform inequality (2.5) into an integral form which will be needed in what follows.

2.6. Lemma: *For $g \in \Sigma$ let G be the inverse of g and suppose G is analytic in $\rho < |\omega| < \infty$. Then for any continuous functions F_1 and F_2 on $|\omega| = r > \rho$,*

$$\begin{aligned} &\Big|\iint\limits_{|w|=|\omega|=r>\rho} F_1(w)F_2(\omega)\frac{G(w)-G(\omega)}{w-\omega}dw\,d\omega\Big|^2 \\ &\leq \sum_{n=0}^{\infty}\Big|\int\limits_{|w|=r} F_1(w)G(w)^{-n}dw\Big|^2 \sum_{m=0}^{\infty}\Big|\int\limits_{|\omega|=r} F_2(\omega)G(\omega)^{-m}d\omega\Big|^2 \end{aligned} \tag{2.7}$$

Proof: We first observe that the two sums on the right side of (2.5) can be written as

$$\sum_{i=0}^{\infty}\sum_{j,s=1}^{n}\alpha_j\bar{\alpha}_s(z_j\bar{z}_s)^{-i}=\sum_{i=0}^{\infty}|\sum_{j=1}^{n}\alpha_j z_j{}^{-i}|^2 \tag{2.8}$$

and

$$\sum_{i=0}^{\infty}\sum_{k,t=1}^{m}\beta_k\bar{\beta}_t(\zeta_k\bar{\zeta}_t)^{-i}=\sum_{i=0}^{\infty}|\sum_{k=1}^{m}\beta_k\zeta_k{}^{-i}|^2, \tag{2.9}$$

respectively. By the assumption all the integrands in (2.7) are continuous, and hence integrable on the respective integral ranges.

For $j=1,2,\ldots,n$, let $\theta_j=2j\pi/n$,

$$\alpha_j=F_1(re^{i\theta_j})(ire^{i\theta_j})\frac{2\pi}{n},\quad \beta_j=F_2(re^{i\theta_j})(ire^{i\theta_j})\frac{2\pi}{n}$$

and

$$z_j=\zeta_j=G(re^{i\theta_j}).$$

Then the double sum on the left side of (2.5) can be written as

$$\sum_{j=1}^{n}\sum_{k=1}^{n}F_1(re^{i\theta_j})F_2(re^{i\theta_k})\frac{G(re^{i\theta_j})-G(re^{i\theta_k})}{re^{i\theta_j}-re^{i\theta_k}}(ire^{i\theta_j})(ire^{i\theta_k})(\frac{2\pi}{n})^2. \tag{2.10}$$

This expression tends to the double integral on the left side of (2.7) as n approaches ∞. Furthermore,

$$\sum_{k=1}^{n}\alpha_k z_k{}^{-j}\longrightarrow\int_{|w|=r}F_1(w)G(w)^{-j}dw$$

and

$$\sum_{k=1}^{n}\beta_k\zeta_k{}^{-j}\longrightarrow\int_{|w|=r}F_2(w)G(w)^{-j}dw$$

as $n\to\infty$. Thus (2.5) yields (2.7) by letting $n\to\infty$. This completes the proof of Lemma 2.6. ∎

3. Inverse coefficients of the class $\Sigma(p)$

By using the integral mean inequality (2.1), Baernstein and Schober [BS] obtained sharp bounds for the inverse coefficients of univalent functions $f \in S(p)$. In this section we make use of both the integral mean inequality (2.4) and the strengthened Grunsky inequality (2.7) to obtain some sharp bounds for the inverse coefficients of univalent functions $g \in \Sigma(p)$, $1 < p < \infty$.

If G is the inverse of a function $g \in \Sigma(p)$, then G admits the expansion (1.1) near $\omega = \infty$ and another expansion

$$G(\omega) = -p + \sum_{n=1}^{\infty} A_n \omega^n \tag{3.1}$$

near $\omega = 0$. In the family $\Sigma(p)$ the function $k_p(\zeta)$, defined in (2.2), plays a similar role as the Koebe function does in the family S. Suppose that its inverse function $K_p(\omega)$ has expansions

$$K_p(\omega) = \omega + \sum_{n=0}^{\infty} K_{p,n} \omega^{-n} \tag{3.2}$$

near $\omega = \infty$ and

$$K_p(\omega) = -p + \sum_{n=1}^{\infty} L_{p,n} \omega^n \tag{3.3}$$

near $\omega = 0$, respectively. Then it follows that

$$\begin{aligned} K_{p,0} &= \lim_{\omega \to \infty} (K_p(\omega) - \omega) \\ &= \lim_{\zeta \to \infty} (\zeta - k_p(\zeta)) = -(p + p^{-1}), \end{aligned} \tag{3.4}$$

and by Cauchy formula

$$\begin{aligned} K_{p,n} &= -\frac{1}{2n\pi i} \oint \omega^n K_p'(\omega) d\omega \\ &= -\frac{1}{2n\pi i} \int_{|\zeta|=r>p} k_p(\zeta)^n d\zeta \\ &= -\frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} p^{n-2k-1} \end{aligned} \tag{3.5}$$

and

$$(3.6)\qquad \begin{aligned} L_{p,n} &= \frac{1}{2n\pi i}\int\limits_{|\zeta|=r<p} k_p(\zeta)^{-n}d\zeta \\ &= -\frac{1}{n}\sum_{k=0}^{\infty}\binom{n+k-1}{n-1}\binom{n+k}{n-1}p^{-(n+2k+1)}. \end{aligned}$$

The main result of this paper is the following.

3.7. THEOREM: *If G is the inverse of a function $g \in \Sigma(p)$ which has expansions (1.1) and (3.1) near $\omega = \infty$ and $\omega = 0$, respectively, then*

$$(3.8)\qquad |B_0| \le |K_{p,0}| = p + p^{-1},$$

and for $n = 1, 2, \dots,$

$$(3.9)\qquad |B_{2n-1}| \le |K_{p,2n-1}|,$$

$$(3.10)\qquad |B_{2n}| \le \sqrt{|K_{p,2n-1}||K_{p,2n+1}}$$

and

$$(3.11)\qquad |A_n| \le |L_{p,n}|.$$

Equality for a single coefficient holds in (3.8), (3.9) or (3.11) only if $g(\zeta) = k_p(\zeta)$.

The proof of Theorem 3.7 depends on the following lemma which is a consequence of Lemma 2.6.

3.12. LEMMA: *Let G be the inverse of $g \in \Sigma$ and let G and G^{-m} have expansions (1.1) and*

$$(3.13)\qquad G(\omega)^{-m} = \sum_{n=m}^{\infty} B_n^{(-m)}\omega^{-n},$$

respectively, near $\omega = \infty$. Then for $n = 1, 2, \dots,$

$$(3.14)\qquad |B_{2n-1}| \le \sum_{m=1}^{n} |B_n^{(-m)}|^2$$

and

$$|B_{2n}|^2 \le \sum_{m=1}^{n} |B_n^{(-m)}|^2 \sum_{m=1}^{n+1} |B_{n+1}^{(-m)}|^2. \tag{3.15}$$

Proof: We may choose $1 < \rho < \infty$ so that $G(\omega)$ is analytic in $\rho < |\omega| < \infty$. Then

$$\frac{G(w) - G(\omega)}{w - \omega} = 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} B_n w^{-i-1} \omega^{i-n}$$

is analytic in $\rho < |w|, |\omega| < \infty$ with respect to each variable. For any integers $j, k \ge 0$, let

$$F_1(w) = w^j \text{ and } F_2(\omega) = \omega^k$$

in (2.7). Then by Cauchy formula, the double integral on the left side of (2.7) reduces to

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{i=0}^{n-1} B_n \iint_{|w|=|\omega|=r>\rho} w^{j-i-1} \omega^{k+i-n} dw d\omega \\ &= 2\pi i \sum_{n=j+1}^{\infty} B_n \int_{|\omega|=r} \omega^{k+j-n} d\omega \\ &= (2\pi i)^2 B_{k+j+1}. \end{aligned}$$

Similarly,

$$\int_{|w|=r} w^j G(w)^{-n} dw = \begin{cases} 2\pi i B_{j+1}^{(-n)}, & \text{if } 0 < n \le j+1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus it follows from (2.7) that

$$|B_{j+k+1}|^2 \le \sum_{m=1}^{j+1} |B_{j+1}^{(-m)}|^2 \sum_{m=1}^{k+1} |B_{k+1}^{(-m)}|^2 \tag{3.16}$$

for all $j, k \ge 0$. Letting $j = k = n - 1$ and letting $j = n - 1$, $k = n$ in (3.16) yield (3.14) and (3.15) , respectively, as desired. ∎

3.17. Remark: The equality in (3.14) holds for any n if $g(\zeta) = k_p(\zeta)$, where $k_p(\zeta)$ is defined in (2.2). To see this we note that (3.14) is derived from (2.7) by letting $F_1(w) = F_2(w) = w^{n-1}$. Thus we only need to show that equality in

(2.7) holds for $g = k_p$ in this particular case. In fact, under change of variables $w = k_p(z)$ and $\omega = k_p(\zeta)$, the double integral on the left side of (2.7) reduces to

$$\sum_{j=0}^{\infty}\Big(\int_{|z|=r'} k_p'(z)k_p(z)^{n-1}z^{-j}dz\Big)^2, \tag{3.18}$$

while both sums on the right side of (2.7) reduce to

$$\sum_{j=0}^{\infty}\Big|\int_{|z|=r'} k_p'(z)k_p(z)^{n-1}z^{-j}dz\Big|^2. \tag{3.19}$$

Since for each j, the integral in (3.18) is a pure imaginary number, the expressions in (3.18) and (3.19) have the same absolute value. Therefore (2.7) holds with equality in this case, and hence the equality in (3.14) holds when $g = k_p$.

3.20. Remark: Inequality (3.14) can also be derived from a Grunsky inequality of functional type, see for example [Sc, (3)], by choosing appropriate functionals on the space of meromorphic functions.

Proof of Theorem 3.7: To prove (3.8) we observe that if $g \in \Sigma(p)$ then $f(z) = g(1/z)^{-1} \in S(1/p)$ and the magnitude of B_0 equals to the magnitude of the second coefficient of the inverse function of f. Thus inequality (3.8) and its sharpness follows from [BS, Theorem 2 (8), n=2].

To prove (3.9) and (3.10), we expand

$$G(\omega)^{-m} = \sum_{n=m}^{\infty} B_n^{(-m)}\omega^{-n}, \tag{3.21}$$

$$\frac{\omega[G(\omega)^{-m}]'}{G(\omega)^{-m}} = -n + \sum_{k=1}^{\infty} M_k^{(-m)}\omega^{-k}, \tag{3.22}$$

$$K_p(\omega)^{-m} = \sum_{n=m}^{\infty} K_n^{(-m)}\omega^{-n}, \tag{3.23}$$

and

$$\frac{\omega[K_p(\omega)^{-m}]'}{K_p(\omega)^{-m}} = -n + \sum_{k=1}^{\infty} C_{p,k}^{(-m)}\omega^{-k} \tag{3.24}$$

near $\omega = \infty$. By Cauchy formula again,

$$\begin{aligned} M_k^{(-m)} &= -\frac{m}{2\pi i}\oint \frac{\omega^k G'(\omega)}{G(\omega)}d\omega \\ &= -\frac{m}{2\pi i}\int_{|\zeta|=r} g(\zeta)^k\zeta^{-1}d\zeta \\ &= -\frac{m}{2\pi}\int_0^{2\pi} g(re^{i\theta})^k d\theta. \end{aligned}$$

Since g is univalent in $\tilde{\Delta}$ and $g(\infty) = \infty$, $g(re^{i\theta})$ is bounded as $r \to 1^+$. Thus letting $r \to 1^+$, the integral mean inequality (2.4) yields

$$\begin{aligned} |M_k^{(-m)}| &\le \frac{m}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^k d\theta \\ &\le \frac{m}{2\pi}\int_0^{2\pi} |k_p(e^{i\theta})|^k d\theta \\ &= \frac{m}{2\pi}\int_0^{2\pi} k_p(e^{i\theta})^k d\theta = |C_{p,k}^{(-m)}| \end{aligned} \tag{3.25}$$

for $k = 1, 2, \ldots$. An argument similar to [BS, p.80] shows that for $k = 1$ we have equality throughout (3.25) only if $g(\zeta) = k_p(\zeta)$. Now we can use the induction method to prove that

$$|B_n^{(-m)}| \le |K_n^{(-m)}| \tag{3.26}$$

for all $n \ge m$. It is easy to see that (3.26) holds for $n = m$ since

$$B_m^{(-m)} = K_m^{(-m)} = 1.$$

Suppose that (3.26) holds for all $n = m+1, m+2, \ldots, N-1$. From the expansions (3.21) and (3.22) it follows that the coefficients $B_n^{(-m)}$ and $M_k^{(-m)}$ satisfy the identity

$$-(n-m)B_n^{(-m)} = \sum_{k=1}^{n-m} M_k^{(-m)} B_{n-k}^{(-m)}.$$

Similarly, $K_n^{(-m)}$ and $C_{p,k}^{(-m)}$ satisfy

$$-(n-m)K_n^{(-m)} = \sum_{k=1}^{n-m} C_{p,k}^{(-m)} K_{n-k}^{(-m)}.$$

Since $C_{p,k}^{(-m)} \le 0$, it follows that $K_n^{(-m)} \ge 0$ for all $n \ge m$. Thus by (3.25) and the assumption on (3.26), we obtain

$$
\begin{aligned}
(N-m)|B_N^{(-m)}| &\le \sum_{k=1}^{N-m} |M_k^{(-m)}||B_{N-k}^{(-m)}| \\
&\le \sum_{k=1}^{N-m} |C_{p,k}^{(-m)}||K_{N-k}^{(-m)}| \\
&= -\sum_{k=1}^{N-m} C_{p,k}^{(-m)} K_{N-k}^{(-m)} = (N-m)|K_N^{(-m)}|.
\end{aligned}
\tag{3.27}
$$

This proves (3.26). Thus it follows from (3.14), (3.15) and Remark 3.17 that

$$|B_{2n-1}| \le \sum_{m=1}^{n} |K_n^{(-m)}|^2 = |K_{p,2n-1}|$$

and

$$|B_{2n}|^2 \le \sum_{m=1}^{n} |K_n^{-m)}|^2 \sum_{m=1}^{n+1} |K_{n+1}^{(-m)}|^2 = |K_{p,2n-1}||K_{p,2n+1}|.$$

This proves the inequalities (3.9) and (3.10).

For $n = 1$ (3.9) reduces to $|B_1| \le 1$. Since $B_1 = -b_1$, the equality in (3.9) holds for $n = 1$ only if $|b_1| = 1$. By the well known area principle for the class Σ, see for example [Du, Theorem 2.1], this can happen only if $g = k_p$. From the process of deriving (3.9), one can see that the equality in (3.9) holds for $n > 1$ only if (3.25) holds with equality for $k = 1$. This can happen only if $g = k_p$.

For the proof of (3.11), we expand the following functions

$$\frac{\omega G'(\omega)}{G(\omega)} = \sum_{n=1}^{\infty} N_n \omega^n \tag{3.28}$$

and

$$\frac{\omega K_p'(\omega)}{K_p(\omega)} = \sum_{n=1}^{\infty} D_{p,n} \omega^n \tag{3.29}$$

near $\omega = 0$. Similar to (3.25) we obtain

$$
\begin{aligned}
|N_n| &\le \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^{-n} d\theta \\
&\le \frac{1}{2\pi} \int_0^{2\pi} |k_p(e^{i\theta})|^{-n} d\theta = D_{p,n}
\end{aligned}
\tag{3.30}
$$

and the equality holds for $n = 1$ only if $g = k_p$. From expansions (3.1) and (3.28), it follows that

$$nA_n = -pN_n + \sum_{i=1}^{n-1} N_i A_{n-i}.$$

Similarly, $L_{p,n}$ and $D_{p,n}$ satisfy

$$nL_{p,n} = -pD_{p,n} + \sum_{i=1}^{n-1} D_{p,i} L_{p,n-i}.$$

By induction method, one can easily show that

$$\begin{aligned} n|A_n| &\le p|N_n| + \sum_{i=1}^{n-1} |N_i||A_{n-i}| \\ &\le pD_{p,n} + \sum_{i=1}^{n-1} |D_{p,i}||L_{p,n-i}| = n|L_{p,n}| \end{aligned}$$

Thus (3.11) follows. Furthermore, the equality in (3.11) can occur for a single n only if it can occur for $n = 1$. And by (3.30) it can happen only if $g(\zeta) = k_p(\zeta)$. This completes the proof of Theorem 3.7. ∎

3.31. Remarks: Inequality (3.10) is asymptotically sharp as $n \to \infty$. To verify this we note that

$$\frac{|K_{p,n+1}|}{|K_{p,n}|} = \frac{\frac{(n!)^2 p^{n-3}}{n+1} \sum_{i=0}^{n} \frac{(n+1)^2 p^{-2i}}{i!(n+1-i)!(i+1)!(n-i)!}}{\frac{(n!)^2 p^{n-1}}{n} \sum_{i=0}^{n-1} \frac{p^{-2i}}{i!(n-i)!(i+1)!(n-i-1)!}}. \tag{3.32}$$

Since $p > 1$, it is easy to see that both sums in (3.32) converge as $n \to \infty$. Thus $|K_{p,n+1}|/|K_{p,n}|$ converges as $n \to \infty$. This proves that

$$\frac{|K_{p,2n}|^2}{|K_{p,2n-1}||K_{p,2n+1}|} \longrightarrow 1$$

as $n \to \infty$. Thus (3.10) is asymptotically sharp. It is still an open problem to obtain the sharp upper bound for $|B_{2n}|$. But it is natural to conjecture that

$$|B_{2n}| \le |K_{p,2n}|.$$

The proof of Theorem 3.7 depends on both the integral mean inequality (2.4) and Lemma 3.12. We point out that the method of Baernstein and Schober

[BS] by which they obtained sharp estimates for the class $S(p)$ does not apply immediately to the class $\Sigma(p)$ which has been considered here.

Obviously the method that is used to establish Theorem 3.7 applies immediately to the class of nonvanishing meromorphic univalent functions which Netanyahu [Ne] considered. But the resulting estimates are sharp only for odd integers and zero.

4. The class S

Let S denote the well known class of all analytic univalent functions $f(z)$ in Δ with the normalization $f(0) = 0$ and $f'(0) = 1$. Then the inverse function $F(w)$ of $f \in S$ has the expansion

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n \tag{4.1}$$

near $w = 0$. Using a variational method, Löwner [Lö] obtained the following sharp estimates.

4.2. THEOREM ([Lö]): *If $F(w)$ is the inverse of a function $f \in S$ and has expansion* (4.1) *near $w = 0$, then*

$$|A_n| \le \frac{(2n)!}{n!(n+1)!} \tag{4.3}$$

for $n = 2, 3, \ldots$. The equality occurs for a single coefficient if and only if $f(z)$ is the Koebe function $k(z) = z/(1-z)^2$ or its rotation.

Alternate proofs have been given by several authors, see for example [SS], [Fi] and [BS]. Here we give another new and elementary proof for this remarkable result.

Proof of Theorem 4.2: For $n = 0, 1, 2, \ldots$, let

$$c_n = \frac{(2n)!}{n!(n+1)!}.$$

Then the inverse function $K(\omega)$ of $k(\zeta) = \zeta + \zeta^{-1} \in \Sigma$ has the expansion

$$K(\omega) = \omega - \sum_{n=0}^{\infty} c_n \omega^{-(2n+1)}$$

near $\omega = \infty$. From the identity

$$\zeta^2 = \omega K(\omega) - 1 = (K(\omega))^2$$

we obtain that

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-i-1}. \tag{4.4}$$

This identity can also be easily verified by induction.

Next we see that if $f(z) \in S$, then

$$g(\zeta) = \frac{1}{f(1/\zeta)} \in \Sigma \text{ and } g(\zeta) \neq 0.$$

Suppose the inverses of f and g have expansions (4.1) and (1.1), respectively. Then from the identity

$$G(w^{-1})F(w) = 1$$

we obtain

$$A_n = -B_0 A_{n-1} - \sum_{i=1}^{n-2} B_i A_{n-i-1}. \tag{4.5}$$

Therefore, by induction, (4.3) follows from (1.3) and the identities (4.4) and (4.5). ∎

4.6. Remark: The above proof suggests an elementary way of deriving sharp bounds for the inverse coefficients of functions defined in Δ from sharp bounds about the corresponding classes of functions defined in $\tilde{\Delta}$. Unfortunately, this process can not be reversed.

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